12th International Congress on Mathematical Education
Program Name XX-YY-zz (pp. abcde-fghij)
8 July – 15 July, 2012, COEX, Seoul, Korea (This part is for LOC use only. Please do not change this part.)

WORKSHOP IN RECOUNTING AND DECIMAL-WRITING

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To deal with the natural fact Many, we totalize. However, there are hidden ways to count and add. This workshop demonstrates the power of recounting made possible by counting in icons before counting in tens. Recounting shows that natural numbers are decimal numbers carrying units. And recounting allows both proportionality and integration to be introduced in grade one.
Keywords: Count, recount, decimal, proportionality, integration, metamatism, per-number.

INTRODUCTION

Enlightenment mathematics was as a natural science exploring the natural fact Many (Kline, 1972) by grounding its abstract concepts in examples, and by using the lack of falsifying examples to validate its theory. But after abstracting the set-concept, mathematics was turned upside down to modern mathematics or 'metamatism', a mixture of 'meta-matics' defining its concepts as examples of abstractions, and 'mathema-tism' true in the library, but not in the laboratory, as e.g. 2+3 = 5, which has countless counterexamples: 2m+3cm = 203 cm, 2weeks+3days = 17 days etc. Being self-referring, this modern mathematics did not need an outside world. However, a self-referring mathematics turned out to be a self-contradiction. With his paradox on the set of sets not belonging to itself, Russell proved that sets implies self-reference and self-contradiction as known from the classical liar-paradox 'this statement is false' being false when true and true when false: If $M = \{A \mid A \notin A\}$, then $M \in M \Leftrightarrow M \notin M$. Likewise Gödel proved that a well-proven theory is a dream since it will always contain statements that can be neither proved nor disproved. In spite of being neither well-defined nor well-proven, mathematics still teaches metamatism creating big problems to math education.

The primary school '10-IS-ten' claim prevents recounting a total in e.g. 5s or 7s, which removes shifting units and proportionality from grade one. Likewise the 'on-top' claim prevents next-to addition in introducing integration. And finally forcing the false identity 23 upon 2.3 tens prevents decimals and fractions to be introduced to describe the unbundled.

The middle school 'proportionality-IS-linearity' claim prevents recounting in different units to produce per-numbers as 3$/5kg. And the neutralization claim prevents equations to be solved in the natural way, by recounting and restacking.

And the high school 'calculus-IS-limits' claim prevents integration to remain adding per-numbers by finding the area under the per-numbers graph as known form primary school's next-to addition and middle school's adding per-numbers as weighted average.

CONTINGENCY RESEARCH UNMASKS CHOICES PRESENTED AS NATURE

Alternatively, mathematics could return to its roots, Many, guided by contingency research uncovering hidden patronization by discovering alternatives to choices presented as nature.

Ancient Greece saw a controversy on democracy between two different attitudes to knowledge represented by the sophists and the philosophers. The sophists warned that to practice democracy, the people must be enlightened to tell choice from nature in order to prevent hidden patronization by choices presented as nature. To the philosophers, patronization was a natural order since to them all physical is examples of meta-physical forms only visible to the philosophers educated at Plato's academy, who therefore should be given the role as natural patronizing rulers (Russell, 1945).

Later Newton saw that a falling apple obeys, not the unpredictable will of a meta-physical patronizor, but its own predictable physical will. This created the Enlightenment: when an apple obeys its own will, people could do the same and replace patronization with democracy.

Two democracies were installed: one in the US still having its first republic; and one in France, now having its fifth republic. German autocracy tried to stop the French democracy by sending in an army. However, a German mercenary was no match to a French conscript aware of the feudal consequence of defeat. So the French stopped the Germans and later occupied Germany. Unable to use the army, the German autocracy instead used the school to stop enlightenment in spreading from France. As counter-enlightenment, Humboldt used Hegel philosophy to create a patronizing line-organized Bildung school system based upon three principles: To avoid democracy, the people must not be enlightened; instead romanticism should install nationalism so the people sees itself as a 'nation' willing to fight other 'nations', especially the democratic ones; and the population elite should be extracted and receive 'Bildung' to become a knowledge-nobility for a new strong central administration replacing the former blood-nobility unable to stop the French democracy.

As democracies, EU still holds on to line-organized education instead of changing to block-organized education as in the North American republics allowing young students to uncover and develop their personal talent through individually chosen half-year knowledge blocks.

In France, the sophist warning against hidden patronization is kept alive in the post-structural thinking of Derrida, Lyotard, Foucault and Bourdieu. Derrida warns against ungrounded words installing what they label, such word should be 'deconstructed' into labels. Lyotard warns against ungrounded sentences installing political instead of natural correctness. Foucault warns against institutionalized disciplines claiming to express knowledge about humans; instead they install order by disciplining both themselves and their subject. And Bourdieu warns against using education and especially mathematics as symbolic violence to monopolize the knowledge capital for a knowledge-nobility (Tarp, 2004).

Thus contingency research does not refer to, but questions existing research by asking 'Is this nature or choice presented as nature?' To prevent patronization, categories should be grounded in nature using Grounded Theory (Glaser et al, 1967), the method of natural research developed in the first Enlightenment democracy, the American, and resonating with Piaget's principles of natural learning (Piaget, 1970).

MATHEMATICS: A NATURAL SCIENCE ABOUT THE NATURAL FACT MANY

To deal with Many, we iconize, bundle and totalize. 1.order counting bundles sticks in icons with five sticks in the five-icon 5 thus making 5 1s to 1 5s. In this way icons are created for numbers until ten needing no icon since 3.order counting in tens has become the standard.

I	II	III	IIII	IIIII	IIIIII	IIIIIII	IIIIIIII	IIIIIIIII
1	2	3	4	5	6	7	8	9

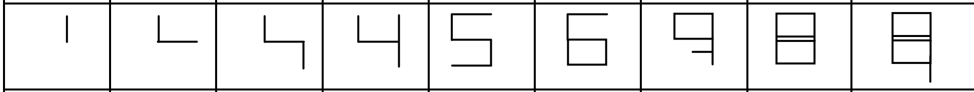


Figure 1. Rearranging sticks in icons transforms 5 1s into 1 5s etc.

2.order counting bundles in icon-bundles. Thus a total of 7 can be bundled in 3s as T = 2 3s & 1 to be placed in a right single-cup and in a left bundle-cup where a bundle can be traded, first to a thick stick representing a bundle glued together, then to a normal stick representing the bundle by being placed in the left bundle-cup. Now the cup-contents is described by icons, first using 'cup-writing' 2)1), then using 'decimal-writing' to separate the left bundle-cup from the right single-cup, and including the unit 3s, T = 2.1 3s.

IIIIIII -> III III I -> III III) I) -> █ █) I) -> II) I) -> 2)1) -> 2.1 3s

Also bundles can be bundled and placed in a new cup to the left. Thus in 6 3s & 2 1s, the 6 3-bundles can be rebundled into two 3-bundles of 3-bundles, i.e. as 2))2 or 2)0)2), leading to the decimal number 20.2 3s: III III) II) -> II)) II), or 6)2) = 2)0)2, thus 6.2 3s = 20.2 3s.

Adding an extra cup to the right shows that multiplying or dividing with the bundle-size just moves the decimal point: T = 2.1 3s = 2)1) -> 2)1)) = 2.1 3s * 3 = 21.0 3s and vice versa.

Operations iconize the bundling and stacking processes. Taking away 4 is iconized as – 4 showing the trace left when dragging away the 4. Taking away 4s is iconized as /4 showing the broom sweeping away the 4s. Building up a stack of 3 4s is iconized as 3x4 showing a 3 times lifting of the 4s. Placing a stack of 2 singles next-to a stack of bundles is iconized as + 2 showing the juxtaposition of the two stacks. And bundling bundles is iconized as ^ 2 showing the lifting away of e.g. 3 3-bundles reappearing as 1 3x3-bundle, i.e. as a 1 3^2-bundle.

Thus soft 'geometric addition' adds bundles next-to, e.g. T = 2 4s + 3 1s, while hard 'algebraic addition' adds on-top, e.g. T = 2 7s + 3 7s = 5 7s. Soft 'geometric subtraction' places part of a stack next to the stack, e.g. 3 4s = 2 4s + 1 4s, while hard 'algebraic subtraction' removes from the top, e.g. T 3 4s – 1 4s = 2 4s. Soft 'geometric multiplication' repeats adding on-top, e.g. T = 2* (3 7s) = 6 7s, while hard 'algebraic multiplication' recounts in tens, making 2* (3 7s) not 6 7s but 4 tens & 2 1s = 4.2 tens = 42. And soft 'geometric division' recounts a stack in a different bundle-size, e.g. 9 = (9/4)*4 = 2*4 + 1, while hard 'algebraic division' recounts in tens, e.g. 9/4 = 2.25 = 0.225 tens.

Numbers and operations can be combined to calculations predicting the counting results. The 'recount-formula' T = T/b bs = (T/b)*b tells that the total T is counted in bs by taking away bs T/b times. Thus recounting a total of T = 3 6s in 7s, the prediction says T = (3*6)/7 7s. Using a calculator we get the result 2 7s and some leftovers that can be found by the 'rest-formula'

T = (T–b) + b telling that T–b is the rest when b is taken away and placed next-to: 3*6 = (3*6–2*7) + 2*7 = 4 + 2*7. So the combined prediction says T = 3*6 = 2*7 + 4*1 = 2.4 7s.

This prediction holds when tested: IIIIII IIIIII IIIIII -> IIIIIII IIIIIII IIII.

A total as T = 2.3 5s can be described as a geometrical number by using squares. After stacking the two 5-bundles, the 3 leftovers may be placed either next to the stack as a stack of 1s, which can be written as 2.3 5s; or on top of the stack of 5-bundles counted as 5s, i.e. as 3 = (3/5)*5 thus giving a total of T = 2 3/5 5s.

Thus 3/5 5s is just another way of writing 0.3 5s. And when bundling in tens, 3/ten tens is the same as 0.3 tens. So for any bundle-number, we define ‘count-fractions’ by 3/b = 0.3. And we see that count-fractions are rooted in bundling the leftovers.

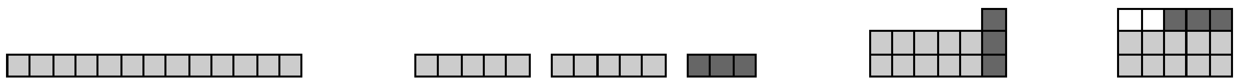


Figure 2. A Total has 3 unbundled added next-to or on-top: T = 2.3 * 5 = 2 3/5 * 5

Count-fractions describe the unbundled 1s when counting in bundles. In the case of bundling in 5s or tens, 4 bundles & 3 unbundled can be written as respectively T = 4.3 5s = 4 3/5 5s or T = 4.3 tens = 4 3/10 tens. In this way both 3/5 of 5 is 3 and 3/10 of 10 is 3. As to 3/5 of 20, 20 can be recounted to 5 fours, and since 3/5 of 5 units is 3 units, 3/5 of 5 4s = 3 4s = 3*4 = 12.

So already in primary school 2.order recounting in icon-bundles enables learners to predict and practice changing units, the leitmotif of mathematics, reappearing later as proportionality and per-numbers. This also allows practicing the scientific method using formulas for predictions to be tested. In this ten-free zone it becomes possible to introduce the core of mathematics using 1digit numbers only (Zybartas et al, 2005). The MATHeCADEMY.net has developed its CATS-approach, Count&Add in Time&Space, to introduce teachers to a grounded approach to mathematics as a natural science investigating the natural fact Many by counting by bundling and stacking, and using recounting at all school levels.

3.ORDER COUNTING IN TENS PREVENTS RECOUNTING

3.order counting in tens should be postponed as long as possible. Before introducing ten as 10, i.e. as the standard bundle-size, 5 should be the standard bundle-size together with a sloppy way of writing numbers hiding both the decimal point and the unit so that e.g. 3.2 5s becomes first 3.2 and then 32, thus introducing place values where the left 3 means 5-bundles and the right 2 means unbundled singles. This leads to the observation that the chosen bundle-size does not need an icon since it is not used when using place values, nor in the counting sequence: 1, 2, 3, 4, Bundle, 1B1, 1B2, 1B3, 1B4, 2B, 2B1, etc.; or 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, etc.

Thus counting in tens can begin using neither a ten-icon nor the ten-name: 8, 9, Bundle, 1Bundle1, 1B2, 1B3, …, 1B9, 2B, 2B1, etc. Then the name bundle can be replaced by the name ten counting 8, 9, Ten, 1Ten1, 1T2, …, 1T9, 2T, 2T1, etc. Finally the sloppy words eleven and twelve can be used meaning ‘1 left’ and ‘2 left’ in ‘Anglish’, i.e. in old English.

A premature introduction of ten as THE standard bundle may make ten a ‘cognitive bomb’.

When beginning to count in tens, numbers are no more written as natural numbers, i.e. as decimals carrying units. Instead numbers are written using the sloppy place-value method hiding both the total and the unit, and misplacing the decimal point: T = 2.3 tens -> plain 23.

Almost all operations change meanings. Soft addition next-to as in T = 2.3 4s = 2*4 + 3*1 is changed to hard addition on-top as in 23 + 48 = 71. Soft multiplication where 3*8 means 3 8s is changed to hard multiplication, i.e. to division recounting the 3 8s in tens: 3*8 IS 24.

Now /4 now means divided in 4, not counted in 4s. Only –3 still means take away 3.

As shown by the recount- and rest-formulas, with 2.order counting the order of operations is: first /, then *, then –, and finally +. With 3.order counting in tens this order is turned around: first +, then –, then *, and finally /.

With ten as the only bundle-size, recounting is impossible to do and to predict by formulas since asking '3 8s = ? tens' leads to T = (3*8/ten)*ten that cannot be calculated on a calculator. Now the answer is given by multiplication, 3*8 = 24 = 2 tens + 4 ones, thus transforming multiplication into division.

PATRONIZING VERSUS GROUNDED MATHEMATICS IN PRIMARY SCHOOL

In primary school the tradition skips 1.order and 2.order counting and goes directly to 3.order counting claiming that '10 IS ten', i.e. the follower of 9 in spite of the fact that 10 is the follower of 4 when counting in 5s. A grounded alternative postpones 3.order counting until after 2.order counting has introduced recounting in different units as an introduction to proportionality; and until after soft addition next-to has introduced integration.

The tradition presents 1digit numbers as symbols and 2digit numbers as natural numbers. A grounded alternative introduces 1digit numbers as what they really are: icons rearranging the sticks they represent; and introduces the natural numbers as what they really are: decimal-numbers with units, e.g. 2.3 4s, and 2.3 tens instead of just 23.

The tradition presents addition on-top as the first of the four operations, where e.g. 7 + 4 = 11 forces the introduction of ten as the bundle-size, and forces the sloppy way of writing 2digit numbers without decimals or units. A grounded alternative first introduces addition next-to so that 2 5s + 4 1s means placing a stack of 4 1s next to a stack of 2 5s, i.e. as 2.4 5s.

The tradition presents hard multiplication as the third operation with tables to be learned by heart, forcing all stacks to be recounted in tens, 3*8 IS 24 etc. A grounded alternative first introduces soft multiplication so that 3*8 means a stack of 3 8s, not needing to be recounted into tens before being recounted in other icon-bundles as 5s or 7s etc.

The tradition presents division as the last of the four operations, where /4 means to split in 4. A grounded alternative introduces division as the first operation, where /4 means counting in 4s. Likewise the recount formula T = (T/b)*b together with a calculator is introduced from grade one to allow formulas and calculators to predict results to be tested by counting.

The tradition introduces 'mathematism' true in the library but not in the laboratory, by teaching that '2 + 3 IS 5' in spite of the fact that 2weeks + 3days = 17days, 2m + 3c = 203cm etc. A grounded alternative always includes the units when adding, e.g. 2 4s + 3 5s = 4.3 5s.

THE WORKSHOP'S LEARNING PRINCIPLES

The workshop builds upon two learning principles. The first says that proper actions prepare individuals to meet the outside world. Wanting to prepare the learner for this meeting, education should be described in terms of actions. Humans deal with the natural fact Many by totalising. Consequently, education should teach learners how to count and how to add, but not how to math since math is not an action word, a verb. Thus counting and adding in time and space are the two core competences to acquire in mathematics education, resonating with the words algebra and geometry meaning to reunite in Arabic and to measure earth in Greek.

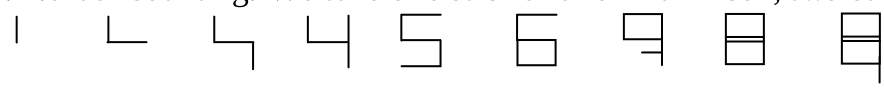
The second learning principle says 'greifen vor begreifen' or 'graps before grasping' or 'through the hands to the head'. Basically it says that categories are names associated with what is grasped by the hands or pointed to. This allows categories to be grounded in hand held objects and observations. This principle resonates with Piaget and with Grounded theory.

A. COUNTING BY BUNDLING AND STACKING, RECOUNTING

In this section we deal with the natural fact Many by performing 1.order counting producing icons and 2.order counting bundling in icons. Thus 3.order counting in tens is postponed.

A0. We place a total of nine sticks on a table. I I I I I I I I I

A1. 1.order counting: We take one stick and form a 1-icon, two sticks to form a 2-icon etc.



A2. 3.order counting in tens: We count fingers in 5s to experience that 5 becomes 10 not needing and icon: 1, 2, 3, 4, B = 10, B1 = 11, B2 = 12, B3 = 13, B4 = 14, B5 = BB = 20, etc. Then we count fingers in 4s, first with whole fingers, then with finger parts. So choosing 3.order counting in ten makes ten to 10 not needing an icon.

A3. 2.order counting in icons: We add 3 sticks and recount the total in 2s, 3s and 4s:

T = 6 2s = 6*2, T = 4 3s = 4*3, T = 3 4s = 3*4.

A4. Cup-counting: We add two sticks and recount the total in 5s, using two cups, a left cup for the bundles and a right cup for the unbundled. We need not put all five sticks in the bundle cup, only 1 stick since 5 1s = 1 5s. Thus we get

T = I I I I I I I I I I I I I I -> IIIII IIIII I I I I -> I I) I I I I)

A5. Cup-writing: Writing down the result, we use parentheses for cups and a decimal point to separate the bundles from the unbundled.

T = I I) I I I I I) -> 2) 4) -> 2.4 so T = 2.4 5s

A6. We recount the total in 6s, 7s, 8s and 9s using cup-writing.

T = I I I I I I I I I I I I I I -> I I) I I) -> 2) 2) -> 2.2 so T = 2.2 6s

Likewise T = 2.0 7s, T = 1.6 8s and T = 1.5 9s.

A7. Overloads: We recount the total in 3s

T = I I I I I I I I I I I I I I -> I I I I) I I) -> 4) 2) -> 4.2 so T = 4.2 3s, or since 3 threes = 1 three-threes

T = I I I I) I I) -> III I) I I) -> I) I) I I) -> 1)1)2) -> 11.2 so T = 11.2 3s

Consequently T = 4.2 3s = 11.2 3s = 1 three-threes & 1 threes & 2 ones.

A8. We recount the total in 2s

T = I I I I I I I I I I I I I I I -> II II II II II II II I-> 7)0) -> 1)5)0) -> 2)3)0) -> 3)1)0) -> 1)1)1)0)

Consequently T = 7 2s = 15 2s = 23 2s = 31 2s = 111 2s

A9. Recounting 8.7 4s, 4 1s in a right cup can be moved to 1 4s in its left cup and vice versa.

T = 8.7 4s = 8)7) = 8+1)-4+7) = 9)3) = 1)5)3) = 2)1)3) = 2)1-1)+4+3) = 2)0)7) = 1)4)7)

Consequently T = 8.7 4s = 9.3 4s = 15.3 4s = 21.3 4s = 20.7 4s = 14.7 4s

A10. We recount 2.3 5s in 6s, 7s, 8s, and 9s

T = 2.3 5s = 2)3) 5s -> 2)1) 6s -> 1)6) 7s -> 1)5) 8s -> 1)4) 9s

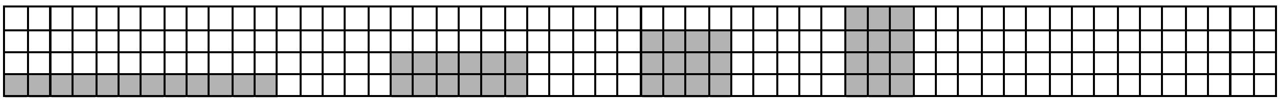
Consequently T = 2.3 5s = 2.1 6s = 1.6 7s = 1.5 8s = 1.4 9s

A11. We recount 2.3 5s in 4s, 3s, 2s, and 1s

T = 2.3 5s = 2)3) 5s -> 2)5) 4s -> 2)7) 3s -> 2)9) 2s -> ?? 1s

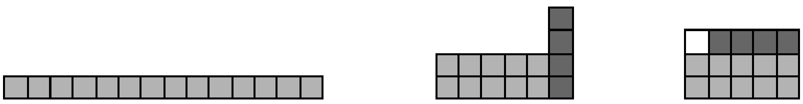
Consequently T = 2.3 5s = 2.5 4s = 2.7 3s = 2.9 2s. T cannot be bundled in 1s since 1^2 = 1.

A12. Grounding fractions. When counting, we use squares instead of sticks to get 'geometric numbers'. We recount 9+3 squares in 6s, 4s and 3s.



T = 2 6s = 2*6 = 3*4 = 4*3

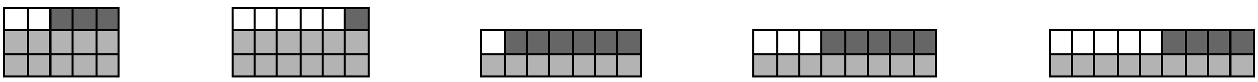
A13. We recount 9+5 squares in 5s



T = 2.4 5s = 2 4/5 5s

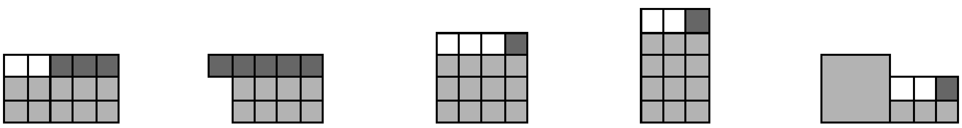
A14. We recount 2 3/5 5s in 6s, 7s, 8s, and 9s:

T = 2 3/5 5s = 2 1/6 6s = 1 6/7 7s = 1 5/8 8s = 1 4/9 9s



2 3/5 5s = 2 1/6 6s = 1 6/7 7s = 1 5/8 8s = 1 4/9 9s

A15. We recount 2.3 5s in 4s and 3s:



2 3/5 5s = 2 5/4 4s = 3 1/4 4s = 4 1/3 3 = 11 1/3 3s

3 5s = 2.3 5s + 2 4 4s = 3.1 4s + 3 12 3s = 11.1 3s + 2

B. INVENTING OPERATIONS TO PREDICT COUNTING RESULTS

In this section we combine numbers with operations to predicting formulas.

B1. The rest formula: From 7 sticks we' take away 3' to be placed next-to.

T = I I I I I I I -> I I I I I I I or 7 = (7-3) + 3 = 4 + 3 or T = (T– b) +b

B2. A calculator predicts. We calculate 8 – 5 on a calculator and perform the action

Prediction: 8 – 5 = 3. Test: I I I I I I I I -> I I I I I I I I or 8 = (8 – 5) + 5 = 3 + 5

B3. The recount formula: From 8 sticks we 'take away 2s' to be placed next-to.

T = I I I I I I I I -> II II II II or 8 = 8/2 2s = (8/2) * 2 = 4 * 2 or T = (T/b) * b = T/b bs

B4. A calculator predicts. We calculate 6/2 on a calculator and perform the action

Prediction: 6/2 = 3. Test: I I I I I I -> II II II or 6 = (6/2) * 2 = 3 * 2

B5. Predicting leftovers: We calculate 7/2 on a calculator and perform the action

Prediction: 7/2 = 3.r and 7 = (7 – 3*2) + 3*2 = 1 + 3*2 = 3.1 2s

Test: I I I I I I I -> II II II I or T = 3.1 2s

C. SOLVING EQUATIONS BY RESTACKING AND REBUNDLING

In this section we look at reversed calculations also called equations and observe that the natural way to solve equations is to move numbers to the opposite side with opposite sign.

C1. We solve the equation x + 5 = 9 by restacking and perform the action

x + 5 = 9 = (9 – 5) + 5, so x = 9 –5 = 4 . Test: **X** I I I I I = I I I I I I I I I = I I I I I I I I I

C2. We solve the equation x * 2 = 6 by rebundling and perform the action

x * 2 = 6 = (6/2)*2, so x = 6/2 = 3 . Test: **XX** = I I I I I I = II II II = III III since 3 2s = 2 3 s.

C3. We solve the equation 2*x + 1 = 7 by restacking and rebundling

2*x + 1 = 7 = (7 – 1) +1 = 6 + 1, so x*2 = 6 = (6/2)*2 = 3*2, so x = (7-1)/2 = 6/2 = 3.

D. SPLITTING STACKS

In this section we look at splitting stacks by taking away, or selling, a part.

D1. From a stack of 4.2 5s is sold 1.4 5s. What is left?

Transform 4.2 5s into cup-writing	T = 4.2 5s = 4) 2)
Change icons into sticks	T = IIII) II)
Change 1 5s into 5 1s	T = III) IIIII II)
Remove the 1.4 5s	T = II) III) + I) IIII)
Go back to icons and decimals	T = 2.3 5s + 1. 4 5s
Test the result by adding	T = 3.7 5s = 4. 2 5s

D2. Splitting, using numbers.

Transform 4.2 5s into cup-writing	T = 4.2 5s = 4) 2)
Change 1 5s into 5 1s	T = 4-1) 5+2) = 3) 7)
Remove the 1.4 5s	T = 2) 3) + 1) 4)
Go back to decimals	T = 2.3 5s + 1.4 5s
Test the result by adding	T = 3.7 5s = 4. 2 5s

D3. From a stack of 42 (4.2 tens) is sold 14 (1.4 tens). What is left?

Transform 42 into cup-writing	T = 42 = 4) 2)
Change 1 tens into 10 1s	T = 4-1) 10+2) = 3) 12)
Remove the 14 = 1.4 tens	T = 2) 8) + 1) 4)
Go back to decimals	T = 2.8 tens + 1.4 tens = 28 + 14
Test the result by adding	T = 2)8) + 1)4) = 3)12) = 4)2)

E. ADDING STACKS

Once counted, stacks can be added on-top or next-to. Adding on-top means the units must be the same. Adding next-to integrates the units.

E1. To a stack of 2.3 5s is bought 1.4 5s. What is the Total?

<table>
<tr><td>Transform 2.3 5s and 1.4 5s into cup-writing</td><td>2.3 5s + 1.4 5s = 2)3) + 1)4)</td></tr>
<tr><td>Change icons into sticks</td><td>T = | |) | | |) + |) | | | |)</td></tr>
<tr><td>Adding 1.4 5s to the 2.3 5s gives 3.7 5s</td><td>T = | | |) | | | | | | |)</td></tr>
<tr><td>Change 1 5s into 5 1s</td><td>T = | | |) ||||| | |) -> | | | |) | |)</td></tr>
<tr><td>Write down the addition result</td><td>2.3 5s + 1.4 5s = 3.7 5s = 4.2 5s</td></tr>
</table>

E2. Adding using numbers. To a stack of 2.3 5s is bought 3.2 4s. What is the total in 4s?

Recount the 2.3 5s in 4s	T = (2*5+3)/4 *4 = 3.1 *4
Add 3.1 4s and 3.2 4s	T = 3.1 4s + 3.2 4s = 6.3 4s
Recount the result	T = 12.3 4s

E3. Add the two stacks 2.3 5s and 3.2 4s as 9s (integration).

Recount the 2.3 5s in 9s	T = (2*5+3)/9 *9 = 1.4 *9
Recount 3.2 4s in 9s	T = (3*4+2)/9 *9 = 1.5 *9
Perform the addition	T = 1.4 9s + 1.5 9s = 2.9 9s
Recount the result	T = 3.0 9s

F. ADDING OR REMOVING CUPS

In this section we see that adding and removing cups to the right means multiplying or dividing with the bundle-number, which just moves the decimal point one place.

Multiply 3.2 5s with 5 by adding a cup to the right	3)2) -> 3)2)) = 3)2) *5
Divide 14.0 5s with 5 by removing a cup to the right	1)4)) -> 1)4) = 1)4)) / 5

G. RECOUNTING IN DIFFERENT PHYSICAL UNITS CREATES PER-NUMBERS

In this section we see how per-numbers as 2$/5kg = 2/5 $/kg are used to relate physical units.

G1. With 2$/5kg, what does 40 kg cost?

We simply recount 40 in 5s: T = 40kg = (40/5)*5kg = (40/5)*2$ = 16$

G2. With 2$/5kg, how much can 12 $ buy?

We simply recount 12 in 2s: T = 12$ = (12/2)* 2$ = (12/2)* 5kg = 30kg

G3. Transform the per-number 2/5 u/u to percent %.

We simply recounts 100 in 5s: T = 100u = (100/5)*5u -> (100/5)*2u = 40u, so 2/5 = 40/100.

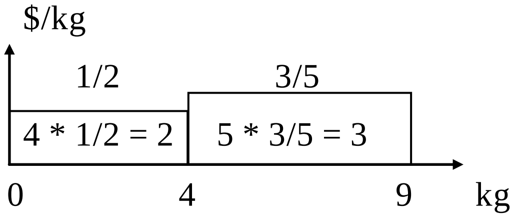
G4. Find 20% of 350$.

We recount the 350 in 100s: T = 350$ = (350/100)*100$ -> (350/100)*20$ = 70$.

H. ADDING PER-NUMBERS

In this section we see that their areas add variable per-numbers, thus rooting integration.

H1. 4 kg at 1/2 $kg + 5kg at 3/5 $/kg = 9 kg at ? $/kg. The answer is given by a graph.

4kg at 1/2 $/kg = 4 * 1/2 = 2 5kg at 3/5 $/kg = 5 * 3/5 = 3	
9kg at ? $/kg = 9 * ? = 5 ? = 5/9	

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